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MACKEY-FUNCTOR STRUCTURE ON THE BRAUER GROUPS OF A FINITE GALOIS COVERING OF SCHEMES

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Abstract. For any finite étale covering of schemes, we can associate two homomorphisms for Brauer groups, namely the pull-back and the norm map. These homomorphisms make Brauer groups into a bivariant functor (a Mackey functor). Restricting to a finite Galois covering of schemes, we obtain a cohomological Mackey functor on its Galois group. This is a generalization of the result for rings by Ford [5]. Applying Bley and Boltje's theorem [1], we can derive certain isomorphisms for the Brauer groups of intermediate coverings.

1. Introduction

In this paper, a scheme S is always assumed to be Noetherian, and $\pi(S)$ denotes its étale fundamental group. Since we use Čech cohomology, we assume S satisfies the following:

Assumption 1.1. For any finite set E of poits of S, there exists an open set $U \subset S$, such that U contains every point in E.

As for the étale fundamental group and related notion, we follow the terminology in [9]. For example a finite étale covering is just a finite étale morphism of schemes.

Our aim is to make the following generalization of the result for rings by Ford [5].

Corollary (Corollary 4.2)**.** Let $\pi : Y \to X$ be a finite Galois covering of schemes with Galois group G. Then the correspondence

$$H \le G \mapsto \mathrm{Br}(Y/H)$$

forms a cohomological Mackey functor on G.

This follows from our main theorem;

Theorem (Theorem 3.5)**.** Let S be a connected Noetherian scheme. Let (FEt/S) denote the category of finite étale coverings over S. Then, the Brauer group functor Br forms a cohomological Mackey functor on (FEt/S).

As in Definition 3.1, a Mackey functor is a bivariant pair of functors $\mathrm{Br} = (\mathrm{Br}^*, \mathrm{Br}_*)$. For any morphism $\pi : Y \to X$, the contravariant part $\mathrm{Br}^*(\pi) : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is the pull-back, and $\mathrm{Br}_*(\pi) : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is the norm map defined later.

By applying Bley and Boltje's theorem (Fact 5.2) to Corollary 4.2, we can obtain certain relations between Brauer groups of intermediate coverings:

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Corollary (Corollary 5.3)**.** Let X be a connected Noetherian scheme and $\pi : Y \to X$ be a finite Galois covering with $\mathrm{Gal}(Y/X) = G$.
(i) Let ℓ be a prime number. If $H \leq G$ is not ℓ-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:\mathrm{odd}}} \mathrm{Br}(Y/U)(\ell)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:\mathrm{even}}} \mathrm{Br}(Y/U)(\ell)^{|U|}.$$

(ii) If $H \leq G$ is not hypoelementary, then there is a natural isomorphism of abelian groups

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:\mathrm{odd}}} \mathrm{Br}(Y/U)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:\mathrm{even}}} \mathrm{Br}(Y/U)^{|U|}.$$

2. Restriction and corestriction

Remark 2.1. For any scheme X, there exists a natural monomorphism

$$\chi_X : \mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X) := H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X})_{\mathrm{tor}}\ ,$$

such that for any morphism $\pi : Y \to X$,

$$\begin{tikzcd} \mathrm{Br}(X) \arrow[r, "\pi^*"] \arrow[d, hook, "\chi_X"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Br}(Y) \arrow[d, hook, "\chi_Y"] \\ H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}) \arrow[r, "\pi^*"] & H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \end{tikzcd}$$

is a commutative diagram.

Here $\pi^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is the pull-back of Azumaya algebras, while $\pi^* : H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}) \to H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y})$ is defined as the composition of the canonical morphism

$$H^2_{\mathrm{et}}(X, \pi_*\mathbb{G}_{m,Y}) \to H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y})$$

and

$$H^2_{\mathrm{et}}(\pi_\sharp) : H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}) \to H^2_{\mathrm{et}}(X, \pi_*\mathbb{G}_{m,Y}),$$

where $\pi_\sharp : \mathbb{G}_{m,X} \to \pi_*\mathbb{G}_{m,Y}$ is the canonical (structural) homomorphism of étale sheaves on X. We call these π^* the *restriction maps*.

Remark 2.2. For any finite étale covering $\pi : Y \to X$, there exists a homomorphism of étale sheaves on X

$$N_{Y/X} : \pi_*\mathbb{G}_{m,Y} \to \mathbb{G}_{m,X}$$

which induces the norm map for finite étale ring extensions.

When $\pi : Y \to X$ is a finite étale covering, the canonical homomorphism $H^2_{\mathrm{et}}(X, \pi_*\mathbb{G}_{m,Y}) \to H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y})$ becomes isomorphic (cf. [6]). By composing $H^2_{\mathrm{et}}(N_{Y/X})$ with the inverse of this canonical isomorphism, we define the *corestriction map* for cohomology groups:

$$\mathrm{cor}_\pi : H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \overset{\cong}{\to} H^2_{\mathrm{et}}(X, \pi_*\mathbb{G}_{m,Y}) \xrightarrow{H^2_{\mathrm{et}}(N_{Y/X})} H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}).$$

Proposition 2.3. *Let $\pi : Y \to X$ as before. There exists a corestriction homomorphism for Brauer groups*

$$\mathrm{cor}_\pi : \mathrm{Br}(Y) \to \mathrm{Br}(X),$$

such that

$$\begin{tikzcd}
\mathrm{Br}(Y) \arrow[r, "\mathrm{cor}_\pi"] \arrow[d, hook, "\chi_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Br}(X) \arrow[d, hook, "\chi_X"] \\
H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \arrow[r, "\mathrm{cor}_\pi"'] & H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X})
\end{tikzcd}$$

is commutative.

To construct $\mathrm{cor} : \mathrm{Br}(Y) \to \mathrm{Br}(X)$, we define a monoidal functor

$$\mathcal{N}_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X).$$

Lemma 2.4. *Let $\pi : Y \to X$ be a finite étale covering of constant degree d. There exists a monoidal functor (unique up to a natural isomorphism)*

$$\mathcal{N}_\pi = \mathcal{N}_{Y/X} : \text{q-Coh}(Y) \to \text{q-Coh}(X),$$

(q-Coh(X) : *the category of quasi-coherent Zariski sheaves on X). such that*
(i) *When Y is isomorphic to a disjoint union of d-copies of X, i.e. when $Y = \coprod_{1\le i\le d} Y_i$ and $\exists \eta_i : X \xrightarrow{\cong} Y_i$, then $\mathcal{N}_{Y/X}$ is defined by*

$$\mathcal{N}_{Y/X}(\mathcal{F}) := \eta_1^*(\mathcal{F}|_{Y_1}) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \eta_d^*(\mathcal{F}|_{Y_d}) \quad (\forall \mathcal{F} \in \text{q-Coh}(Y)).$$

(ii) *For any pull-back by a morphism $f : X' \to X$*

$$\begin{tikzcd}
Y' \arrow[r, "\pi'"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "f"] \\
Y \arrow[r, "\pi"'] & X
\end{tikzcd}$$

there exists a natural isomorphism of monoidal functors

$$\mathcal{N}_{Y'/X'} \circ g^* \xrightarrow{\cong} f^* \circ \mathcal{N}_{Y/X}.$$

Proof. When Y is isomorphic to a disjoint union of d-copies of X, then $\mathcal{N}_{Y/X}$ is defined by as in (i).

For a general case, remark that

Remark 2.5. For any finite étale covering $\pi : Y \to X$ of constant degree d, there exists a fpqc morphism $f : X' \to X$ such that $Y \times_X X'$ is isomorphic to a disjoint union of d-copies of X'.

$$\begin{tikzcd}
Y' \cong \coprod_d X' \arrow[r, "\pi'"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "f"] \\
Y \arrow[r, "\pi"'] & X
\end{tikzcd}$$

For any $\mathcal{F} \in \text{q-Coh}(Y)$, put $\overline{\mathcal{F}} := \mathcal{N}_{Y'/X'}(g^*(\mathcal{F}))$. Then $\overline{\mathcal{F}}$ descends to yield $\mathcal{N}_{Y/X}(\mathcal{F}) \in \text{q-Coh}(X)$. Thus we obtain a monoidal functor $\mathcal{N}_{Y/X}$. This construction does not depend on the choice of f, up to an isomorphism of monoidal functors. By the reduction to the disjoint-union case as above, we can show (ii). $\square$

While this $\mathcal{N}_{Y/X}$ is a generalization of the norm functor for a finite étale ring extension (Knus-Ojanguren [8], Ferrand [4]), it is also possible to define $\mathcal{N}_{Y/X}$ by gluing those for affines.

Lemma 2.6. *Let $\pi : Y \to X$ be a finite étale covering of constant degree d. $\mathcal{N}_{Y/X}$ has the following properties*:
(0) *$\mathcal{N}_{Y/X}$ is monoidal.*
(1) *For any $\mathcal{F}, \mathcal{G} \in$ q-Coh(Y), there exists a functorial morphism*

$$\theta_{Y/X} : \mathcal{N}_{Y/X}(\mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G})) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{N}_{Y/X}(\mathcal{F}), \mathcal{N}_{Y/X}(\mathcal{G})).$$

(1^+) *Moreover if $\mathcal{G}$ is locally free of finite rank, this is an isomorphism.*
(2) *There exists a natural isomorphism*

$$\mathcal{N}_{Y/X}(\mathcal{O}_Y^{\oplus n}) \cong \mathcal{O}_X^{\oplus n^d}.$$

(2^+) *More generally, if $\mathcal{F}$ is locally free $\mathcal{O}_Y$-module of finite rank n, then $\mathcal{N}_{Y/X}(\mathcal{F})$ becomes locally free $\mathcal{O}_X$-module of rank n^d.*

For a general (non-constant degree) $\pi : Y \to X$, we can define the norm functor on each connected component of X as above, and glue them to obtain the norm functor $\mathcal{N}_{Y/X}$: q-Coh$(Y) \to$ q-Coh(X).

Proof. Conditions (0) and (2) follow from the definition of $\mathcal{N}_{Y/X}$. By taking an affine cover $X = \bigcup_{i \in I} U_i$, ($2^+$) reduces to the case where X, Y are affine, shown by Ferrand [4]. As for condition (1), existence of $\theta_{Y/X}$ simply follows from the fact that $\mathcal{N}_{Y/X}$ is a monoidal functor between closed symmetric monoidal categories. (1^+) is shown by a reduction to the affine case. □

Proof. (proof of Proposition) By the above lemma, especially we have an isomorphism

$$\mathcal{N}_{Y/X}(\mathcal{M}_n(\mathcal{O}_Y)) \cong \mathcal{M}_{n^d}(\mathcal{O}_X)$$

of $\mathcal{O}_X$-algebras, for any finite étale covering Y/X of constant degree d.

Remark that for any $\mathcal{O}_Y$-algebra $\mathcal{A}$ of finite type, $\mathcal{A}$ is an Azumaya algebra if and only if $\mathcal{A}$ is étale locally isomorphic to $\mathcal{M}_n(\mathcal{O}_Y)$. Thus for any Azumaya algebra $\mathcal{A}$, there exists a covering $\mathcal{V} := \{V_i \xrightarrow{g_i} Y\}_{i \in I}$ of Y in the étale topology (simply written '$\mathcal{V} \in \mathrm{Cov}_{\mathrm{et}}(Y)$') such that

$$g_i^* \mathcal{A} \cong \mathcal{M}_{n_i}(\mathcal{O}_{V_i}) \quad (\exists n_i \in \mathbb{N}).$$

Replacing $\mathcal{V}$ by its refinement, we may assume that there exists a covering $\mathcal{U} = \{U_i \xrightarrow{f_i} X\}_{i \in I} \in \mathrm{Cov}_{\mathrm{et}}(X)$ such that

$$\mathcal{V} = \pi^* \mathcal{U} := \{Y \times_X U_i \xrightarrow{\mathrm{pr}_Y} Y\}_{i \in I}.$$

Then we have $f_i^* \mathcal{N}_{Y/X}(\mathcal{A}) \cong \mathcal{N}_{V_i/U_i}(g_i^* \mathcal{A}) \cong \mathcal{M}_{n_i^d}(\mathcal{O}_{U_i})$. Thus $\mathcal{N}_{Y/X}(\mathcal{A})$ also becomes an Azumaya algebra.

By the isomorphism

$$\mathcal{N}_{Y/X}(\mathcal{E}\mathrm{nd}(\mathcal{E})) \cong \mathcal{E}\mathrm{nd}(\mathcal{N}_{Y/X}(\mathcal{E})) \quad (\forall \mathcal{E}\text{: locally free of finite rank})$$

and the monoidality of $\mathcal{N}_{Y/X}$, we obtain a well-defined homomorphism

$$\begin{array}{rccc} \mathrm{cor}_\pi : & \mathrm{Br}(Y) & \longrightarrow & \mathrm{Br}(X) \\ & \Psi & & \Psi \\ & \mathcal{A} & \longmapsto & \mathcal{N}_{Y/X}(\mathcal{A}). \end{array}$$

By using Čech cohomology, we can show the commutativity of

$$\begin{tikzcd} \mathrm{Br}(Y) \arrow[r, "\mathrm{cor}_\pi"] \arrow[d, hook, "\chi_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Br}(X) \arrow[d, hook, "\chi_X"] \\ H^2_{\mathrm{et}}(Y, \mathbb{G}_{m,Y}) \arrow[r, "\mathrm{cor}_\pi"'] & H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}). \end{tikzcd}$$

□

3. Brauer-group Mackey functor

For any profinite group G, let (fin. G-space) denote the category of finite discrete G-spaces and equivariant G-maps.

Definition 3.1. Let $\mathcal{C}$ be a Galois category, with fundamental functor F (i.e. there exists a profinite group $\pi(\mathcal{C})$ such that F gives an equivalence from $\mathcal{C}$ to (fin. $\pi(\mathcal{C})$-space)).

A cohomological Mackey functor on $\mathcal{C}$ is a pair of functors $M = (M^*, M_*)$ from $\mathcal{C}$ to Ab, where M^* is contravariant and M_* is covariant, satisfying the following conditions:
(0) $M^*(X) = M_*(X)(=: M(X)) \quad (\forall X \in \mathrm{Ob}(\mathcal{C}))$.
(1) (Additivity) For each coproduct $X \overset{i_X}{\hookrightarrow} X \coprod Y \overset{i_Y}{\hookleftarrow} Y$ in $\mathcal{C}$, the canonical morphism

$$(M^*(i_X), M^*(i_Y)) : M(X\textstyle\coprod Y) \rightarrow M(X) \oplus M(Y)$$

is an isomorphism.
(2) (Mackey condition) For any pull-back diagram

$$\begin{tikzcd} Y' \arrow[r, "\varpi'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\pi'"] \\ X' \arrow[r, "\varpi"'] & X \end{tikzcd} ,$$

the following diagram is commutative:

$$\begin{tikzcd} M(Y) \arrow[r, "M^*(\varpi')"] \arrow[d, "M_*(\pi)"'] \arrow[dr, phantom, "\circlearrowleft"] & M(Y') \arrow[d, "M_*(\pi')"] \\ M(X) \arrow[r, "M^*(\varpi)"'] & M(X') \end{tikzcd}$$

(3) (Cohomological condition) For any morphism $\pi : X \to Y$ in $\mathcal{C}$ with X and Y connected, we have

$$M_*(\pi) \circ M^*(\pi) = \text{multiplication by } \deg(\pi)$$

where $\deg(\pi) := \sharp F(Y)/\sharp F(X)$.

$$\begin{tikzcd} & M(Y) \arrow[dr, "M_*(\pi)"] \arrow[d, phantom, "\circlearrowleft"] & \\ M(X) \arrow[ur, "M^*(\pi)"] \arrow[rr, bend right, "\deg \pi"'] & {} & M(X) \end{tikzcd}$$

A standard example is the cohomological Mackey functor on a profinite group G (in terminology of [1], a cohomological Mackey functor on the finite natural Mackey system on G):

Definition 3.2. Let G be a profinite group, and put $\mathcal{C} :=$ (fin. G-space), $F :=$ id. A cohomological Mackey functor on $\mathcal{C}$ is simply called a cohomological Mackey functor on G, and their category is denoted by $Mack_c(G)$.

Remark 3.3. Since any object X in (fin. G-space) is a direct sum of transitive G-sets of the form G/H where H is a open subgroup of G, a Mackey functor on G is equivalent to the following datum:
An abelian group $M(H)$ for each open $H \leq G$, with structure maps
- a homomorphism $\mathrm{res}^H_K : M(H) \to M(K)$ for each open $K \leq H \leq G$,
- a homomorphism $\mathrm{cor}^H_K : M(K) \to M(H)$ for each open $K \leq H \leq G$,
- a homomorphism $c_{g,H} : M(H) \to M({}^gH)$ for each open $H \leq G$ and $g \in G$,

where ${}^gH := gHg^{-1}$, satisfying certain compatibilities (cf. [1]). Here $M(G/H)$ is abbreviated to $M(H)$ for any open $H \leq G$.

Example 3.4. In this notation, for any G-module A and any $n \geq 0$, the group cohomology

$$H \mapsto H^n(H, A) \quad (\forall H \leq G \text{ open})$$

becomes a cohomological Mackey functor on G, with appropriate structure maps.

For any finite étale covering $\pi : Y \to X$, put $\mathrm{Br}^*(\pi) := \mathrm{res}_\pi$ and $\mathrm{Br}_*(\pi) := \mathrm{cor}_\pi$. Then we obtain a cohomological Mackey functor Br (and similarly Br', $H^2_{\mathrm{et}}(-, \mathbb{G}_m)$):

Theorem 3.5. *For any connected Noetherian scheme S, we have a sequence of cohomological Mackey functors on* (FEt$/S$)

$$\mathrm{Br} \hookrightarrow \mathrm{Br}' \hookrightarrow H^2_{\mathrm{et}}(-, \mathbb{G}_m).$$

Proof. We only show Mackey and cohomological conditions. Since restrictions and corestrictions are compatible with inclusions

$$\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X) \hookrightarrow H^2_{\mathrm{et}}(X, \mathbb{G}_{m,X}),$$

it suffices to show for $H^2_{\mathrm{et}}(-, \mathbb{G}_m)$.
(Mackey condition) For any pull-back diagram

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & Y' \arrow[l, "\varpi'"'] \arrow[d, "\pi'"] \\ X & X' \arrow[l, "\varpi"] \end{tikzcd}$$

in (FEt$/S$), we have a commutative diagram

$$\begin{tikzcd} \pi_*\mathbb{G}_{m,Y} \arrow[r, "\pi_*(\varpi'_\sharp)"] \arrow[dd, "N_{Y/X}"'] \arrow[ddrr, phantom, "\circlearrowright"] & \pi_*\varpi'_*\mathbb{G}_{m,Y'} \arrow[dr, phantom, "\cong"] & \\ & & \varpi_*\pi'_*\mathbb{G}_{m,Y'} \arrow[d, "\varpi_*(N_{Y'/X'})"] \\ \mathbb{G}_{m,X} \arrow[rr, "\varpi_\sharp"'] & & \varpi_*\mathbb{G}_{m,X'} \end{tikzcd}$$

This yields a commutative diagram

$$\begin{tikzcd} H^2_{\mathrm{et}}(Y,\mathbb{G}_{m,Y}) \arrow[r, "\mathrm{res}_{\varpi'}"] \arrow[d, "\mathrm{cor}_\pi"'] \arrow[dr, phantom, "\circlearrowright"] & H^2_{\mathrm{et}}(Y',\mathbb{G}_{m,Y'}) \arrow[d, "\mathrm{cor}_{\pi'}"] \\ H^2_{\mathrm{et}}(X,\mathbb{G}_{m,X}) \arrow[r, "\mathrm{res}_{\varpi}"'] & H^2_{\mathrm{et}}(X',\mathbb{G}_{m,X'}) \end{tikzcd} .$$

(Cohomological condition) For any morphism $\pi : Y \to X$ in (FEt/S) with X and Y connected, since

$$N_{Y/X} \circ \pi_\sharp : \mathbb{G}_{m,X} \to \mathbb{G}_{m,X}$$

is equal to the multiplication by $d = \deg(\pi)$

$$\begin{tikzcd} \mathbb{G}_{m,X} \arrow[r, "\pi_\sharp"] \arrow[rr, bend right, "d"'] & \pi_*\mathbb{G}_{m,Y} \arrow[r, "N_{Y/X}"] \arrow[d, phantom, "\circlearrowright"] & \mathbb{G}_{m,X} \\ & {} & \end{tikzcd} ,$$

we obtain $\mathrm{cor}_\pi \circ \mathrm{res}_\pi = d$.

$$\begin{tikzcd} & H^2_{\mathrm{et}}(Y,\mathbb{G}_{m,Y}) \arrow[dr, "\mathrm{cor}_\pi"] & \\ H^2_{\mathrm{et}}(X,\mathbb{G}_{m,X}) \arrow[ur, "\mathrm{res}_\pi"] \arrow[r, "H^2_{\mathrm{et}}(\pi_\sharp)"'] \arrow[rr, bend right, "d"'] & H^2_{\mathrm{et}}(X,\pi_*\mathbb{G}_{m,Y}) \arrow[u, "\cong"', "\mathrm{can.}"] \arrow[r, "H^2_{\mathrm{et}}(N_{Y/X})"'] & H^2_{\mathrm{et}}(X,\mathbb{G}_{m,X}) \end{tikzcd}$$

□

4. Restriction to a finite Galois covering

Thus we have obtained a Mackey functor Br on (FEt/S). By pulling back by a quasi-inverse $\mathcal{S}$ of the fundamental functor

$$F : (\mathrm{FEt}/S) \xrightarrow{\simeq} (\text{fin. } \pi(S)\text{-space}),$$

we can obtain a Mackey functor on $\pi(S)$:

Corollary 4.1. *There is a sequence of cohomological Mackey functors*

$$\mathrm{Br} \circ \mathcal{S} \hookrightarrow \mathrm{Br}' \circ \mathcal{S} \hookrightarrow H^2_{\mathrm{et}}(-,\mathbb{G}_m) \circ \mathcal{S}$$

on $\pi(S)$, *where* $\mathrm{Br} \circ \mathcal{S} := (\mathrm{Br}^* \circ \mathcal{S}, \mathrm{Br}_* \circ \mathcal{S})$ *and so on.*

Corollary 4.2. *Let X be a connected Noetherian scheme. For any finite Galois covering $\pi : Y \to X$ with $\mathrm{Gal}(Y/X) = G$, there exists a cohomological Mackey functor $\mathcal{B}\mathrm{r}$ on G which satisfies*

$$\mathcal{B}\mathrm{r}(H) \cong \mathrm{Br}(Y/H) \quad (\forall H \le G),$$

with structure maps induced from restrictions and corestrictions of Brauer groups. (We abbreviate $\mathcal{B}\mathrm{r}(G/H)$ to $\mathcal{B}\mathrm{r}(H)$.)

Proof. By the previous corollary, we have a cohomological Mackey functor $\mathrm{Br}\circ\mathcal{S}$ on $\pi(X)$. Since there is a projection $\mathrm{pr}:\pi(X)\twoheadrightarrow G^{\mathrm{op}}$, we can regard any finite G^{op}-set naturally as a finite $\pi(X)$-space, to obtain a functor

$$(\text{fin. } G^{\mathrm{op}}\text{-space})\to(\text{fin. } \pi(X)\text{-space}).$$

Pulling back by this functor, and taking the opposite Mackey functor,
we obtain

$$\begin{array}{ccccc} \mathrm{Mack}_c(\pi(X)) & \longrightarrow & \mathrm{Mack}_c(G^{\mathrm{op}}) & \xrightarrow{\mathrm{op}} & \mathrm{Mack}_c(G) \\ \cup\!\shortmid & & & & \cup\!\shortmid \\ M & & \longmapsto & & M_G\,. \end{array}$$

In terms of subgroups of G, M_G satisfies

$$M_G(H)=M(\mathrm{pr}^{-1}(H^{\mathrm{op}}))\quad(\forall H\le G).$$

Applying this to $\mathrm{Br}\circ\mathcal{S}$, we obtain $\mathcal{B}\mathrm{r}:=(Br\circ\mathcal{S})_G\ \in\mathrm{Mack}_c(G)$. Since the equivalence $\mathcal{S}:(\text{fin. }\pi(X)\text{-space})\xrightarrow{\simeq}(\mathrm{FEt}/X)$ satisfies

$$\mathcal{S}(\pi(X)/\mathrm{pr}^{-1}(H^{\mathrm{op}}))\cong Y/H,$$

we have

$$\mathcal{B}\mathrm{r}(H)\cong\mathrm{Br}(Y/H).$$

□

Similarly we can define $\mathcal{B}\mathrm{r}'$ (and also $(H^2_{\mathrm{et}}(-,\mathbb{G}_m)\circ\mathcal{S})_G$). Since $\mathrm{Mack}_c(G)$ is an abelian category with objectwise (co-)kernels (see for example [3]), we can take the quotient Mackey functor $\mathcal{B}\mathrm{r}'\,/\,\mathcal{B}\mathrm{r}\in\mathrm{Mack}_c(G)$, which satisfies

$$(\mathcal{B}\mathrm{r}'\,/\,\mathcal{B}\mathrm{r})(H)\cong(\mathrm{Br}'(Y/H))/(\mathrm{Br}(Y/H)).$$

5. Application of Bley and Boltje's theorem

Let ℓ be a prime number. For any abelian group A, let

$$A(\ell):=\{m\in A\mid\exists e\in\mathbb{N}_{\ge0},\ell^e m=0\}$$

be the ℓ-primary part. This is a $\mathbb{Z}_\ell$-module.

Definition 5.1 ([1])**.** For any finite group H,
H is *ℓ-hypoelementary* $\underset{\mathrm{def}}{\Leftrightarrow}$ H has a normal ℓ-subgroup with a cyclic quotient.
H is *hypoelementary* $\underset{\mathrm{def}}{\Leftrightarrow}$ H is ℓ-hypoelementary for some prime ℓ.

Fact 5.2 ([1])**.** Let M be a cohomological Mackey functor on a finite group G.
(i) Let ℓ be a prime number. If $H\le G$ is not *ℓ-hypoelementary*, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H\\ n:\mathrm{odd}}} M(U)(\ell)^{|U|}\cong\bigoplus_{\substack{U=H_0<\cdots<H_n=H\\ n:\mathrm{even}}} M(U)(\ell)^{|U|}.$$

(ii) If $H\le G$ is not *hypoelementary* and $M(U)$ is torsion for any subgroup $U\le H$, then there is a natural isomorphism of abelian groups

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H\\ n:\mathrm{odd}}} M(U)^{|U|}\cong\bigoplus_{\substack{U=H_0<\cdots<H_n=H\\ n:\mathrm{even}}} M(U)^{|U|}.$$

Here, $|U|$ denotes the order of U.

Applying this theorem to $\mathcal{B}\mathrm{r}$, we obtain the following relations for the Brauer groups of intermediate étale coverings:

Corollary 5.3. *Let X be a connected Noetherian scheme and $\pi : Y \to X$ be a finite Galois covering with* $\mathrm{Gal}(Y/X) = G$.
(i) *Let ℓ be a prime number. If $H \leq G$ is not ℓ-hypoelementary, then there is a natural isomorphism of $\mathbb{Z}_\ell$-modules*

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:odd}} \mathrm{Br}(Y/U)(\ell)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:even}} \mathrm{Br}(Y/U)(\ell)^{|U|}.$$

(ii) *If $H \leq G$ is not hypoelementary, then there is a natural isomorphism of abelian groups*

$$\bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:odd}} \mathrm{Br}(Y/U)^{|U|} \cong \bigoplus_{\substack{U=H_0<\cdots<H_n=H \\ n:even}} \mathrm{Br}(Y/U)^{|U|}.$$

Finally, we derive some numerical equations related to Brauer groups from Corollary 5.3.

Definition 5.4. Let G be a finite group. For any subgroups $U \leq H \leq G$, put

$$\mu(U,H) := \sum_{U=H_0<\cdots<H_n=H} (-1)^n, \quad \text{Möbius function.}$$

If m (resp. m_ℓ) is an additive invariant of abelain groups (resp. $\mathbb{Z}_\ell$-modules) which is finite on Brauer groups, we obtain the following equations:

Corollary 5.5. *Let $\pi : Y \to X$ as before, $G = \mathrm{Gal}(Y/X)$.*
(i) *If $H \leq G$ is not ℓ-hypoelementary,*

$$\sum_{U \leq H} |U| \cdot \mu(U,H) \cdot m_\ell(\mathrm{Br}(Y/U)(\ell)) = 0.$$

(ii) *If $H \leq G$ is not hypoelementary,*

$$\sum_{U \leq H} |U| \cdot \mu(U,H) \cdot m(\mathrm{Br}(Y/U)) = 0.$$

For a prime ℓ and an abelian group A, its corank is defined as $\mathrm{rank}_{\mathbb{Z}_\ell}(T_\ell(A))$, where $T_\ell(A) = \varprojlim_n Ker(\ell^n : A \to A)$. In this note, we denote this by

$$\mathrm{rk}_\ell(A) := \mathrm{rank}_{\mathbb{Z}_\ell}(T_\ell(A)).$$

$\mathrm{Br}(X)(\ell)$ is known to be of finite corank, for example in the following cases ([7]):
- (C1) k: a separably closed or finite field, X: of finite type $/k$, and proper or smooth $/k$, or $\mathrm{char}(k) = 0$ or $\dim X \leq 2$.
- (C2) X: of finite type $/Spec(\mathbb{Z})$, and smooth $/Spec(\mathbb{Z})$ or proper over $\exists$open $\subset Spec(\mathbb{Z})$.

Remark that if Y/X is a finite étale covering and X satisfies (C1) or (C2), then so does Y.

Example 5.6. Assume X satisfies (C1) or (C2). For any non-ℓ-hypoelementary subgroup $H \leq G$, we have an equation

$$\sum_{U \leq H} |U|\mu(U,H) \cdot \mathrm{rk}_\ell(\mathrm{Br}(Y/H)(\ell)) = 0.$$

Another example is related with the comparison of Br and Br′. By Gabber's lemma, for any finite étale covering Y/X, we have

$$\mathrm{Br}'(X)/\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(Y)/\mathrm{Br}(Y).$$

In particular, if $\mathrm{Br}(Y) \subset \mathrm{Br}(Y)'$ is of finite index, then so is $\mathrm{Br}(X) \subset \mathrm{Br}(X)'$.

Example 5.7. Assume X satisfies $[\mathrm{Br}'(Y) : \mathrm{Br}(Y)] < \infty$. Then for any non-hypoelementary subgroup $H \leq G$, we have an equation

$$\sum_{U \leq H} |U|\mu(U,H) \cdot [\mathrm{Br}'(Y/U) : \mathrm{Br}(Y/U)] = 0.$$

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